אינפי 1

פרק 22 - תרגילים מתקדמים נוספים (הפרק באנגלית)

תוכן העניינים

Convergence of a Sequence, Monotone Sequences (סדרות)

Questions

1) Let A be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

2) Let A be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in A such that $|x_0 - a_n| \to d(x_0, A)$. Recall that $d(x, A) = \inf\{|x-a| : a \in A\}$.

3) Let (a_k) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$. Show that the sequence (x_n) converges.

Cauchy Criterion, Bolzano - Weierstrass Theorem

4) Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \to \infty$.

5) Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a subsequence converging to x_0. Show that $x_n \to x_0$.

6) Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer n is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).

a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.

b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.

c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

Continuity and Limits (גבולות ורציפות)

Questions

1) Let $\lim_{x\to 0}\frac{f(x)}{x^2}=5$. Show that $\lim_{x\to 0}\frac{f(x)}{x}=0$.

2) Let $f:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$. Suppose $\lim_{x\to x_0} f(x)$ exists.

Show that $\lim_{x\to 0} f(x+x_0)=\lim_{x\to x_0} f(x)$.

3) Let $f(x)=|x|$ for every $x\in\mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

4) Let $f:[0,\pi]\to\mathbb{R}$ be defined by $f(0)=0$ and $f(x)=x\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ for $x\neq 0$. Is f continuous?

5) Let $[\cdot]$ denote the integer part function and $f:[0,\infty)\to\mathbb{R}$ be defined by $f(x)=[x^2]\sin\pi x$.

a) Show that f is continuous at each $x\neq\sqrt{n},\ n\in\mathbb{N}$. [Here $\mathbb{N}$ includes 0]

b) Show that f is continuous at each $x=k\in\mathbb{N}$.

c) Show that f is discontinuous at each $x=\sqrt{n},\ n\in\mathbb{N}$ such that $x\notin\mathbb{N}$.

6) Let the function $f:[0,1]\to[a,b]$ be one-one and onto. Suppose f is continuous. Show that f^{-1} is also continuous.

7) Let $f:(0,1)\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p,q\in\mathbb{N} \text{ and } p,q \text{ have no common factor}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

a) Suppose $x_n\to x_0$ for some x_0, with $x_n\neq x$ for all $n\in\mathbb{N}$, and suppose $x_n=\frac{p_n}{q_n}\in(0,1)$ where $p_n,q_n\in\mathbb{N}$ have no common factors. Show that $\lim_{n\to\infty} q_n=\infty$.

b) Show that f is continuous at every irrational.

c) Show that f is discontinuous at every rational.

Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

Questions

1) Give an example of a function f on $[0,1]$ which is not continuous but satisfies the IVP*. *We say that f has the property IVP [Intermediate Value Property] on $[a,b]$ if for every $x, y \in [a,b]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x,y]$, such that $f(x_0) = \alpha$.

2) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is a constant function if

a) $f(x)$ is rational for each $x \in \mathbb{R}$.

b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.

3) Let $p(x):\mathbb{R} \to \mathbb{R}$ be a polynomial function of odd degree. Show that p is onto.

4) Let $f,g:[0,1] \to \mathbb{R}$ be continuous such that
$\inf\{f(x): x \in [0,1]\} = \inf\{g(x): x \in [0,1]\}$.
Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

5) A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.

6) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function.

a) Suppose f attains each of values exactly two times. Given:
$f(x_1) = f(x_2) = \alpha$ for some $x_1, x_2, \alpha \in \mathbb{R}$, and $f(x_0) > \alpha$ for some $x_0 \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.

b) Using (a) show that f cannot attain each of its values exactly two times.

Mean Value Theorem, L'Hôpital's Rule, Differentiability (משפט לגראנז', כלל לופיטל וגזירות)

Questions

1) Does there exist a differentiable function $f:[0,2]\to\mathbb{R}$ satisfying $f(0)=-1,\ f(2)=4$ and $f'(x)\le 2$, for all $x\in[0,2]$?

2) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for some $\alpha\in\mathbb{R}$, $|f'(x)|\le\alpha<1$ for all $x\in\mathbb{R}$. Let $a_1\in\mathbb{R}$ and define a sequence (a_n) recursively by $a_{n+1}=f(a_n)$. Show that (a_n) converges.

3) Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $\alpha\in\mathbb{R}$ be such that $f'(a)<\alpha<f'(b)$. Define $g(x)=f(x)-\alpha x$ for all $x\in[a,b]$.

a) Show that there exists $c\in[a,b]$ such that $g'(c)=0$.
Hint: prove by contradiction, noting that $g'(a)<0$ and $g'(b)<0$.

b) From the above, conclude that if a function f is differentiable on an interval $[a,b]$, then f' has the Intermediate Value Property on $[a,b]$.

4) Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $\int_0^1 f(t)dt=1$.

a) Show that there exists $c\in(0,1)$ such that $f(c)=1$.

b) Show that there exist $c_1\ne c_2$ in $(0,1)$ such that $f(c_1)+f(c_2)=2$.

5) Let $f:[0,1]\to\mathbb{R}$ be such that $|f'(x)|<10$ for all $x\in(0,1)$ and let (x_n) be a sequence in $(0,1)$ satisfying the Cauchy criterion. Show that the sequence $(f(x_n))$ converges.

6) Let $f:[0,1]\to\mathbb{R}$ and $a_n=f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right),\ n=1,2,\ldots$
Show that:

a) if f is continuous, then $\sum_{n=1}^{\infty}a_n$ converges;

b) if f is differentiable and $|f'(x)|<\frac{1}{2}\ \forall x\in[0,1]$, then $\sum_{n=1}^{\infty}a_n(\cos n)\sqrt{n}$ converges.

7) Let $p(x)=a+bx+cx^2$. Find all values of $a,b,c\in\mathbb{R}$ for which the function $p(|x|)$ is differentiable at 0.

Power Series, Taylor Series (טורי חזקות וטורי טיילור)

Questions

1) Let $f:(a,b)\to\mathbb{R}$ be infinitely differentiable and let $x_0\in(a,b)$. Suppose that there exists $M>0$ such that $\left|f^{(n)}(x)\right|\le M^n$ for all $n\in\mathbb{N}$ and $x\in(a,b)$. Show that Taylor's series of f around x_0 converges to $f(x)$ for all $x\in(a,b)$.

2) Let (a_n) be a sequence of nonnegative reals and suppose that $(a_n^{\frac{1}{n}})$ is a bounded sequence. For each n, define $A_n=\sup\{a_k^{\frac{1}{k}}:k\ge n\}$.
(A_n) converges since it is decreasing and bounded below (by 0).
So $A_n\to L$ for some $L\ge 0$.

a) Show that if $L<1$, the series $\sum_{n=1}^{\infty}a_n$ converges and if $L>1$ the series diverges.

b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty}a_nx^n$ is $\frac{1}{L}$.